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Entropy and Similarity Measures of q-Rung Orthopair Fuzzy Soft Sets and their Applications in Decision Making Problems

Aparna Sivadas[1] **and Sunil Jacob John**[1,*]

[1]Department of Mathematics, National Institute of Technology Calicut, Calicut-673601, Kerala, India;

aparna$_p$190067ma@nitc.ac.in; sunil@nitc.ac.in.

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Abstract

The q-rung orthopair fuzzy set can represent a wide range of uncertainty in information. When combined with a soft set, the resulting notion of a q-rung orthopair fuzzy soft set (OFSS_q) is more effective in dealing with uncertainties as it allows for parameterization. The OFSS_q is a parameterized family of q-rung orthopair fuzzy sets and a generalization of the Intuitionistic fuzzy soft set (IFSS), the Pythagorean fuzzy soft set (PFSS) and the Fermatean fuzzy soft set (FFSS). While entropy and similarity measures have been defined for these fuzzy set extensions, defining these measures for OFSS_q provides generalized expressions that apply to all these special cases. This article proposes distinct expressions for entropy and similarity measures for OFSS_qs. The proposed entropy measure aids in assessing uncertainty within an OFSS_q, while the similarity measure identifies the degree of similarity between any two OFSS_qs. This article showcases the use of the suggested entropy and similarity measures in decision making, highlighting their effectiveness. Both concepts of entropy and similarity will be applied to decision making problems related to Covid-19 pandemic, especially when some dubious inputs are present, and a quick decision must be made..

Keywords: q-Rung orthopair fuzzy soft set; entropy; similarity measure; decision making.

Corresponding Author:Sunil Jacob John

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1|Introduction

The concept of fuzzy sets was developed as a way to mathematically represent the uncertainty often associated with ideas or phenomena that lack well-defined boundaries. L.A.Zadeh [1] proposed the mathematical definition of a fuzzy set in 1965 and explained its semantic interpretations. Fuzzy sets effectively quantify the linguistic aspects of available facts in decision making problems. Atanassov [2] extended this idea by introducing the intuitionistic fuzzy set (IFS), which includes both membership and non-membership degrees, offering a more comprehensive depiction of uncertainty. Membership and non-membership degrees quantify the positive and negative aspects of the represented idea, respectively. Each evaluation pair in an IFS is known as an orthopair. In later research, Atanassov and Gargov [3] gave interval estimates for the evaluations of membership and non-membership, finding that the evaluations could not always be expressed as orthopairs. The IFS concept evolved into models like the Pythagorean fuzzy set (PFS) [4], the Fermatean fuzzy set (FFS) [5], and the more general q-rung orthopair fuzzy set (OFSq) [6], based on the premise that non-membership should be evaluated independently of membership. Other categories of fuzzy set extensions include neutrosophic sets [7] and their extensions, such as picture fuzzy sets [8] and spherical fuzzy sets [9]. Additional extensions include hesitant fuzzy sets [10], fuzzy multi-sets [11], and plithogenic sets [12]. Recent literature on the theory and applications of fuzzy set extensions can be found in [13, 14, 15, 16].

Molodtsov [17] introduced the concept of soft set as an advanced mathematical tool for managing uncertainty. The central idea of soft set theory is that concepts based on 'belongingness' can be extended by making them dependent on a set of parameters. Research in soft set theory demonstrated that the concept of soft set can be integrated with other established concepts, such as fuzzy sets and IFSs. A soft set is a parameterized collection of sets that can be extended into different hybrid structures, including fuzzy soft set (FSS), intuitionistic fuzzy soft set (IFSS), and Pythagorean fuzzy soft set (PFSS). Insights into various hybrid structures of soft sets are found in articles [18, 19, 20, 21, 22, 23, 24]. The combination of OFS$_q$s with soft sets has emerged as a practical framework in fuzzy mathematics and decision making contexts. Hussain *et al.* [25] initiated the study on q-rung orthopair fuzzy soft sets (OFSS$_q$), devising various mean aggregation operators for OFSS$_q$s and discussed their characteristics. Zulqarnain *et al.* [26] developed geometric aggregation operators using Einstein operational laws and discussed some of their properties. Articles [27, 28, 29, 30] provide relevant literature related to OFSS$_q$s and their generalizations.

Information measures are key concepts used in multi-attribute decision making (MADM) problems across various fields, including pattern recognition, clinical diagnosis, and personnel selection. Several information measures, such as distance, similarity, entropy, and inclusion measures, have been developed to assist in these decision making processes. Extensions of fuzzy sets have been widely applied in recent years to address MADM problems, making the study of information measures of these fuzzy set extensions essential. Several recent articles discuss different information measures for various fuzzy extensions, including those in [31, 32, 33]. Fuzzy entropy measures the amount of fuzzy information obtained from a fuzzy set or system. Based on Shannon's entropy, De Luca and Termini [34] proposed an axiomatic foundation for the entropy of a fuzzy set. Kosko [35] used the ratio of distances of a fuzzy set from its farthest and nearest crisp sets to define fuzzy entropy, while Liu [36] utilized the similarity measure between a fuzzy set and its complement for the same. In 1996, Burillo and Bustince [37] put forward the axiomatic definition for entropy of IFSs. Another set of axioms for the same has been presented by Szmidt and Kacprzyk [38]. These axiomatic structures were both of non-probabilistic type. Peng and Yang [39] constructed axiomatic definitions for information measures of PFSS such as entropy, similarity measures and inclusion measures, also exploring their transformation relationships. Kashyap *et.al* [40] designed a trigonometric entropy measure for PFSSs and utilized it in an MADM based on COPRAS approach. Jiang *et al.* [41] provided an axiomatic definition of intuitionistic entropy for IFSSs, while Athira *et al.* [42] studied entropy measures of PFSSs. Since their introduction, OFSS$_q$s have been widely studied in the literature, with numerous contributions to their advancement. Recently, Ahammad [43] included in his article some novel entropy measures within the framework of OFSS$_q$s to relate more complicated data and to allow decision-makers greater flexibility in managing a variety of decision making situations.

Similarity measure is a key concept in fuzzy sets and their extensions, serving as a mathematical function to assess the similarity between such sets. Various authors suggested different similarity measures for fuzzy sets [44, 45, 46]. Numerous similarity measures between IFSs and PFSs have been created so far. For instance, Olgun *et al.* [47] created a cosine similarity measure for IFSs based on the Choquet integral, whereas Duan and Li [48]

introduced a novel intuitionistic fuzzy similarity measure. Huang *et al.* [49] proposed a similarity metric for IFSs involving the area transformation of an isosceles right triangle. Recently, Patel *et al.* [50] constructed a novel intuitionistic fuzzy similarity measure and used it for a new face recognition method. For PFSs, Arora and Naithani [51] presented similarity measures based on logarithmic functions. Ejegwa [52] improved Zhang and Xu's distance measures for PFSs and their use in pattern recognition. Recent studies [53, 54, 55] have proposed several additional similarity measures. Later, Peng and Liu [56] developed an axiomatic definition of the similarity measure for OFS$_q$s alongside those for entropy, distance measure, and inclusion measure, focusing on creating transformation formulas for these information measures. In [57], the authors attempted to establish similarity measures for OFS$_q$s using cosine and cotangent functions. Ganie [58] constructed similarity measures using t-norms for OFS$_q$s. Recently, Sivadas *et al.* [59] studied distance and similarity measures of (p, q)-fuzzy sets [60], which generalize OFS$_q$s. In the case of IFSSs, Muthukumar and Krishnan [61] studied the idea of similarity measure between two IFSSs. Niher Das and Lohani [62] presented a Sugeno integral-based similarity measure for IFSSs. Saika *et al.* [63] developed cosine simlarity measures for IFSSs and applied them in creating an evidence-based model for handling the dynamics of a Chikungunya problem. Lately, Athira *et al.* [64] studied the similarity measures of PFSSs and effectively applied them in clustering algorithms.

The literature review indicates that OFSS$_q$s are adaptable and capable of addressing problems with uncertainty. Entropy measures are crucial in the analysis of fuzzy sets and their hybrid structures, as they play a significant role in quantifying uncertain information present in the data. Since, OFSS$_q$s are superior tools for managing uncertainty compared to IFSSs and PFSSs, determining their entropy is also pertinent. It has been found that entropy can be applied to decision making problems, thereby expanding their applicability across various domains. The axioms for an entropy measure of the OFSS$_q$s were introduced in [43], This article presents a slightly modified set of axioms for an entropy measure in the context of OFSS$_q$s, incorporating a more general monotonicity condition. However, any entropy measure that satisfies the axioms in [43] will also satisfy the modified set of axioms. The revised set of axioms is designed to more accurately reflect the fuzziness of the OFSS$_q$s, focusing on the distance between the membership and non-membership values of the associated OFS$_q$s. Similarity measures of OFSS$_q$s are not yet found in the existing literature. Since the OFSS$_q$ adds a parameter to the standard OFS$_q$ concept, it has wide practical applications. To apply OFSS$_q$s effectively , it is crucial to develop appropriate similarity measures that accurately reflect the similarity between these sets. Similarity measures are key in expanding theories and proposing various practical applications, such as MADM, pattern recognition, and medical diagnosis. Hence, this study aims to develop similarity measures for OFSS$_q$s. Also, the article explains the construction of a set of similarity measures for OFSS$_q$s, using fuzzy equivalences and aggregation functions. Although, the similarity measures developed in this article are straightforward extensions of existing measures to OFSS$_q$s, they are governed by the parameter q, which can be adjusted at any moment to handle inputs effectively in decision making problems. The article provides mathematical expressions developed for both entropy and similarity measures. The concept of entropy will be applied to a decision making problem related to a surveillance system at a checkpoint during a pandemic, especially when some dubious inputs are present while a quick decision must be made. Similarity measures are utilized for the preliminary classification of patients based on symptoms, guiding further specialized tests when a large number of people seek healthcare facilities during a pandemic.

The rest of this work is organized as follows: Section 2 comprises some basic concepts related to OFSS$_q$s. Section 3 depicts the axioms for entropy and similarity measures of OFSS$_q$s. Section 4 incorporates the application of the proposed measures in decision making problems.

2|Preliminary Concepts

This section reviews the concepts from the literature. Orthopair fuzzy sets are fuzzy sets where the membership grades of each element in the universal set are a pair of values in the unit interval: one value denotes the degree of membership, and the other denotes the degree of non-membership. An orthopair fuzzy set is termed 'never-full' if a membership value of 1 implies a non-membership value of 0, and vice versa, for each element in the universal set. The following definition refers to a specific orthopair fuzzy set called the q-rung orthopair fuzzy set (OFS$_q$). Note that throughout this article, X denotes a finite universal set and P a finite parameter set.

Definition 1. [6] *Let* X *be a universal set. Let* $q \geq 1$*, a q-rung orthopair fuzzy set (*OFS_q*) on* X *is given by*

$$F = \{\langle x, u_F(x), v_F(x)\rangle \mid x \in \mathrm{X}\}$$

where $u_F, v_F : \mathrm{X} \to [0,1]$ *denote the membership and non-membership functions of* F *respectively and* $u_F(x)$*,* $v_F(x)$ *satisfy* $0 \leq (u_F(x))^q + (v_F(x))^q \leq 1$ *for each* $x \in \mathrm{X}$*.*

When $q = 1$, F represents an intuitionistic fuzzy set [2]; for $q = 2$ it becomes a Pythagorean fuzzy set [4] and for $q = 3$ it is labelled as a Fermatean fuzzy set [5].

Recall that, in the case of intuitionistic fuzzy sets, the condition $0 \leq (u_F(x)) + (v_F(x)) \leq 1$ is imposed. While this restriction is logically sound, it presents a significant risk of errors. In practice, contradictions in data can occur where $0 \leq (u_F(x)) + (v_F(x)) > 1$. Although such occurrences are much rarer, they become significantly problematic when dealing with large real-life datasets, consisting of millions of entries, where strict adherence to $0 \leq (u_F(x)) + (v_F(x)) \leq 1$ cannot be guaranteed. Such instances in real-life datasets justifies the proposition of OFS_qs.

Definition 2. [17] *Let* X *be a universal set,* P *be a set of parameters,* $\mathbb{P}(\mathrm{X})$ *be the power set of* X*. A pair* (F, P) *is called a soft set over* X *where* F *is a mapping from* P *to* $\mathbb{P}(\mathrm{X})$*.*

The q-rung orthopair fuzzy soft set (OFSS_q) is a combination of OFS_q and soft set, and is mathematically defined as in the following definition.

Definition 3. [25] *Let* X *be a universal set,* P *be a set of parameters, and* $OFS_q(\mathrm{X})$ *be the set of all q-rung orthopair fuzzy sets on* X*. A pair* (F, P) *is a q-rung orthopair fuzzy soft set (*$OFSS_q$*) on* X*, where* $F : P \to OFS_q(\mathrm{X})$*.*

Remark 1. *A q-rung orthopair fuzzy soft set* (F, P) *can be represented as*

$$(F, P) = \{(p, \{\langle x, u_{F(p)}(x), v_{F(p)}(x)\rangle : x \in \mathrm{X}\}) : p \in P\}.$$

It can be represented as a matrix, with each entry being an orthopair.

As mentioned in [65], for a never-full orthopair fuzzy set A on a finite set, it is possible to find an index $q_0 \geq 1$, which is the smallest index for which A is an OFS_q for $q \geq q_0$. Let $A = \{(x_i, u_i, v_i) : x_i \in X\}$ be a never-full orthopair fuzzy set. Then, for each x_i, $i = 1, 2, \ldots n$, find q_i such that $v_i \leq \sqrt[q]{1 - u_i{}^q}$ for $q \geq q_i$, and then $q_0 = \max_i q_i$. Given a finite collection of never-full orthopair fuzzy sets for a finite set of parameters, it is possible to find an optimum index, such that for all q greater than or equal to this optimum index such that, the collection forms a q-rung orthopair fuzzy soft set. This optimum index is the maximum of the smallest indices of the constituent never-full orthopair fuzzy sets.

Definition 4. [25] *Let* X *be a universal set and* P *be a set of parameters. Let* $(F_1, P) = \{(p, \{\langle x, u_{F_1(p)}(x), v_{F_1(p)}(x)\rangle : x \in \mathrm{X}\}) : p \in P\}$*and* $(F_2, P) = \{(p, \{\langle x, u_{F_2(p)}(x), v_{F_2(p)}(x)\rangle : x \in \mathrm{X}\}) : p \in P\}$ *be two* $OFSS_q$*s on* X*. The following are some operations, such as union, intersection, and complement, along with the subset relation:*

(1) $(F_1, P) \subseteq (F_2, P)$*, if* $u_{F_1(p)}(x) \leq u_{F_2(p)}(x)$ *and* $v_{F_1(p)}(x) \geq v_{F_1(p)}$ *for* $x \in \mathrm{X}$*,* $p \in P$*.*

(2) $(F_1, P) \cup (F_2, P) = \{(p, \{\langle x, \max\{u_{F_1(p)}(x), u_{F_2(p)}(x)\}, \min\{v_{F_1(p)}(x), v_{F_2(p)}(x)\}\rangle : x \in \mathrm{X}\}) : p \in P\}$

(3) $(F_1, P) \cap (F_2, P) = \{(p, \{\langle x, \min\{u_{F_1(p)}(x), u_{F_2(p)}(x)\}, \max\{v_{F_1(p)}(x), v_{F_2(p)}(x)\}\rangle : x \in \mathrm{X}\}) : p \in P\}$

(4) $(F_1, P)^c = \{(p, \{\langle x, v_{F_1(p)}(x), u_{F_1(p)}(x)\rangle : x \in \mathrm{X}\}) : p \in P\}$.

3|Entropy and Similarity Measures for q-rung orthopair fuzzy soft sets

3.1. Entropy Measures for q-rung orthopair fuzzy soft sets

Every fuzzy set can be viewed as a classical soft set. Since fuzzy soft sets are a fuzzy extension of classical soft sets, each fuzzy set can also be seen as a fuzzy soft set. In other words, fuzzy soft sets can be considered as a generalization of fuzzy sets. As OFSS_qs are extensions of fuzzy soft sets, the study of entropy measures, which quantify the fuzziness of OFSS_qs is appropriate. The axiomatic definition of an entropy measure for OFSS_qs were initially introduced in [43]. This study aims to present a revised set of axioms for defining entropy measures in the framework of OFSS_qs. Since an OFSS_q is a collection of OFS_qs, its entropy depends on the entropy of the constituent OFS_qs. From the literature on the properties of entropy measures defined for IFSSs, PFSSs and OFS_qs, to define a measure of fuzziness for OFSS_qs, a real-valued function from the set of all OFSS_qs on the universal set to $[0, 1]$ with the following specific characteristics is considered.

(1) The entropy of an OFSS_q is minimum when it reduces to a soft set.

(2) The entropy of an OFSS_q is maximum when the image of each parameter, which is an OFS_q, has the maximum fuzziness. According to [56], the entropy of an OFS_q is maximum when the membership and non-membership values are equal for each element in the universal set.

(3) The entropy of an OFSS_q and its complement are equal.

(4) The entropy of an OFSS_q increases with an increase in fuzziness of each of the constituent OFS_qs. The closeness of membership and non-membership values in an OFS_qs given by $\Delta = |u - v|$ determines the fuzziness of each OFS_q. As Δ decreases, fuzziness increases and vice versa.

Definition 5. *Let* X *be a universal set,* P *be a set of parameters, and* $OFSS_q(\mathrm{X}, P)$ *denotes the set of all q-rung orthopair fuzzy soft sets on* X *with parameter set* P. *A entropy measure of an* $OFSS_q$ *is defined as a function* $E : OFSS_q(\mathrm{X}, P) \to [0, 1]$ *satisfying all the following axioms for all* (F, P) *and* $(G, P) \in OFSS_q(\mathrm{X}, P)$

(1) $E(F, P) = 0$ *if and only if* (F, P) *is a soft set*

(2) $E(F, P) = 1$ *if and only if* $u_{F(p)}(x) = v_{F(p)}(x)$ *for all* $p \in P$ *and* $x \in \mathrm{X}$.

(3) $E(F, P) = E((F, P)^c)$

(4) $E(G, P) \leq E(F, P)$ *if* $\mid u_{G(p)}(x) - v_{G(p)}(x) \mid > \mid u_{F(p)}(x) - v_{F(p)}(x) \mid$ *for all* $p \in P$ *and* $x \in \mathrm{X}$.

The monotonicity condition in the definition of entropy mentioned in [43] is

$$E(G,P) \leq E(F,P) \,\text{if} \begin{cases} u_{G(p)}(x) \leq u_{F(p)}(x) \,\text{and}\, v_{G(p)}(x) \geq v_{F(p)}(x) \,\text{for}\, u_{F(p)}(x) \leq v_{F(p)}(x) \\ \text{or} \\ u_{G(p)}(x) \geq u_{F(p)}(x) \,\text{and}\, v_{G(p)}(x) \leq v_{F(p)}(x) \,\text{for}\, u_{F(p)}(x) \geq v_{F(p)}(x) \end{cases} \tag{1}$$

for all $p \in P$ and $x \in \mathrm{X}$.

Equation (1) verifies the monotonicity of the entropy candidate function only when the membership and non-membership values satisfy one of two specified conditions. However, these conditions do not fully capture the fuzziness inherent in OFS_qs within the image set of an OFSS_q. Similarly, in the axiomatic definition of entropy measures in the context of hybrid structures like IFSS [41] and PFSS [42], the monotonicity axiom is validated using conditions which do not generically represent the fuzziness of the respective hybrid structure. A similar axiom as that of equation (1) is presented in [66] for the entropy measure of PFSSs. To establish more robust entropy measures for OFSS_qs, the monotonicity of the entropy candidate function must be guaranteed under a general condition that adequately reflects the fuzziness of individual OFS_qs. Thus, the article introduces a revised set of axioms for an entropy measure within the framework of OFSS_qs (see Axiom 4 of Definition 5).

Theorem 1. *Let* X *be a universal set with n elements $\{x_1, x_2 \dots x_n\}$ and P be a set of parameters, $P = \{p_1, p_2, \dots p_m\}$. Then*

$$E(F,P) = 1 - \sqrt[q]{\frac{1}{\mid X \mid\mid P \mid}\sum_{j=1}^{m}\sum_{i=1}^{n}|u_{F(p_j)}(x_i) - v_{F(p_j)}(x_i)|^q} \tag{2}$$

is an entropy measure for $OFSS_q$s.

Proof: (1) Suppose (F,P) is a soft set, that is, each parameter in P is mapped to a crisp subset of the universal set X. For a crisp subset of X expressed as a q-ROFS, the membership grade of each $x \in$ X is either $(0,1)$ or $(1,0)$. Hence

$u_{F(p_j)}(x_i) = 0, v_{F(p_j)}(x_i) = 1$ or $u_{F(p_j)}(x_i) = 1, v_{F(p_j)}(x_i) = 0$ for $i = 1,2,....n$ and $j = 1,2,....m$.

In either of the cases

$E(F,P) = 1 - \sqrt[q]{\frac{1}{mn}mn} = 0.$

Conversely suppose $E(F,P) = 0$.

$1 - \sqrt[q]{\frac{1}{|X||P|}\sum_{j=1}^{m}\sum_{i=1}^{n}|u_{F(p_j)}(x_i) - v_{F(p_j)}(x_i)|^q} = 0$

$\sum_{j=1}^{m}\sum_{i=1}^{n}|u^q_{F(p_j)}(x_i) - v^q_{F(p_j)}(x_i)|^q = mn$

Since the sum of mn terms is mn, $|u^q_{F(p_j)}(x_i) - v^q_{F(p_j)}(x_i)|^q = 1$ for all i and j.

That is $(u_{F(p_j)}(x_i), v_{F(p_j)}(x_i))$ is either $(1,0)$ or $(0,1)$ for all i and j, which turns (F,P) to a soft set.

(2) Suppose $u_{F(p_j)}(x_i) = v_{F(p_j)}(x_i)$ for all $p_j \in P$ and $x_i \in$ X, then

$E(F,P) = 1 - \sqrt[q]{\frac{1}{|X||P|} \times 0} = 1$

Conversely suppose $E(F,P) = 1$

$\sqrt[q]{\frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}|u_{F(p_j)}(x_i) - v_{F(p_j)}(x_i)|^q} = 0$

Thus $u_{F(p_j)}(x_i) = v_{F(p_j)}(x_i)$ for all i, j.

(3) $E(F,P) = E((F,P)^c)$, evident from the suggested expression.

(4) If $\mid u_{G(p_j)}(x_i) - v_{G(p_j)}(x_i) \mid > \mid u_{F(p_j)}(x_i) - v_{F(p_j)}(x_i) \mid$ then

$\mid u_{G(p_j)}(x_i) - v_{G(p_j)}(x_i) \mid^q > \mid u_{F(p_j)}(x_i) - v_{F(p_j)}(x_i) \mid^q$ as $q \geq 1$

$\frac{1}{|X||P|}\sum_{j=1}^{m}\sum_{i=1}^{n}|u_{G(p_j)}(x_i) - v_{G(p_j)}(x_i)|^q \geq \frac{1}{|X||P|}\sum_{j=1}^{m}\sum_{i=1}^{n}|u_{F(p_j)}(x_i) - v_{F(p_j)}(x_i)|^q$

$\sqrt[q]{\frac{1}{|X||P|}\sum_{j=1}^{m}\sum_{i=1}^{n}|u_{G(p_j)}(x_i) - v_{G(p_j)}(x_i)|^q} \geq \sqrt{\frac{1}{|X||P|}\sum_{j=1}^{m}\sum_{i=1}^{n}|u_{F(p_j)}(x_i) - v_{F(p_j)}(x_i)|^q}$

$$1-\sqrt[q]{\frac{1}{|X||P|}\sum_{j=1}^{m}\sum_{i=1}^{n}|u_{G(p_j)}(x_i)-v_{G(p_j)}(x_i)|^q} \leqslant 1-\sqrt[q]{\frac{1}{|X||P|}\sum_{j=1}^{m}\sum_{i=1}^{n}|u_{F(p_j)}(x_i)-v_{F(p_j)}(x_i)|^q}$$

Thus $E(G,P) \leq E(F,P)$.

□

Even though the entropy measure proposed in Theorem 1 is a straightforward extension of measures used for other hybrid fuzzy structures, it satisfies all the axioms of Definition 1, including the modified monotonicity condition.

Example 1. *Consider a data set on a universal set* $X = \{x_1, x_2, x_3, x_4, x_5\}$ *with a parameter set* $P = \{p_1, p_2, p_3, p_4\}$

	p_1	p_2	p_3	p_4
x_1	$(0.6, 0.3)$	$(0.8, 0.4)$	$(0.9, 0.5)$	$(0.9, 0.4)$
x_2	$(0.8, 0.2)$	$(0.5, 0.1)$	$(0.7, 0.6)$	$(0.8, 0.3)$
x_3	$(0.9, 0.2)$	$(0.7, 0.3)$	$(0.5, 0.4)$	$(0.7, 0.4)$
x_4	$(0.7, 0.4)$	$(0.8, 0.6)$	$(0.9, 0.2)$	$(0.7, 0.2)$
x_5	$(0.8, 0.6)$	$(0.9, 0.4)$	$(0.7, 0.1)$	$(0.9, 0.4)$

Since the given data set contains never-full orthopair fuzzy set for each parameter, it is possible to find the optimum indices for each parameter p_i.

p_i	q_0
p_1	2
p_2	2
p_3	3
p_4	2

Thus the optimum q_0 *for the given data set is* $q_0 = \max\{2, 2, 3, 2\} = 3$, *which means the data set is an* $OFSS_q$ *for* $q \geq 3$, *denoted as* (F, P). *By Theorem* 1,

$$E(F,P) = 1-\sqrt[3]{\frac{1}{5\times 4}\sum_{j=1}^{4}\sum_{i=1}^{5}|u_{F(p_j)}(x_i)-v_{F(p_j)}(x_i)|^3} = 0.5284$$

3.2. Similarity Measures for q-rung orthopair fuzzy soft sets

The similarity measure is an information measure used to determine the extent of similarity between given sets. In other words, it assesses the closeness between pairs of sets. There are situations where it is necessary to determine whether two patterns or images are identical, approximately identical, or at least to what degree they are identical. This is where similarity measures come into play, and they have extensive applications in several areas such as pattern recognition, image processing, region extraction, psychology, handwriting recognition, decision making, coding theory, and more. However, expressions for similarity measures in the context of OFSS_qs have not yet appeared in the literature. This subsection presents the similarity metric defined within the OFSS_q environment and suggests a construction method to develop them.

Definition 6. *Let* X *be a universal set and* P *be a set of parameters. Let* $OFSS_q(\mathrm{X}, P)$ *denotes the set of all q-rung orthopair fuzzy soft sets on* X *with parameter set* P. *A similarity measure is a mapping* $S : OFSS_q(\mathrm{X}, P) \times OFSS_q(\mathrm{X}, P) \to [0, 1]$ *satisfying all the following axioms for* $(F, P), (G, P), (H, P) \in OFSS_q(\mathrm{X}, P)$.

(1) $0 \leq S((F, P), (G, P)) \leq 1$

(2) $S((F, P), (G, P)) = S((G, P), (F, P))$

(3) $S((F, P), (G, P)) = 1$ *if and only if* $(F, P) = (G, P)$

(4) *If* $(F,P) \subseteq (G,P) \subseteq (H,P)$ *then* $S((F,P),(H,P)) \leq S((F,P),(G,P))$ *and* $S((F,P),(H,P)) \leq S((G,P),(H,P))$.

Theorem 2. *Let* $\mathrm{X} = \{x_1, x_2 \ldots x_n\}$ *be a universal set and* $P = \{p_1, p_2, ...p_m\}$ *be a set of parameters. Then*

$$S((F,P),(G,P)) = 1 - \sqrt[q]{\frac{1}{2 \mid \mathrm{X} \mid\mid P \mid} \sum_{j=1}^{m} \sum_{i=1}^{n} (\mid u_{F(p_j)}(x_i) - u_{G(p_j)}(x_i) \mid^q + \mid v_{F(p_j)}(x_i) - v_{G(p_j)}(x_i) \mid^q)} \tag{3}$$

is a similarity measure for $OFSS_q$*s.*

Proof: (1) Since $u_{F(p_j)}(x_i)$, $v_{F(p_j)}(x_i)$, $u_{G(p_j)}(x_i)$, $v_{F(p_j)}(x_i) \in [0,1]$ we have

$$\mid u_{F(p_j)}(x_i) - u_{G(p_j)}(x_i) \mid \text{ and } \mid v_{F(p_j)}(x_i) - v_{G(p_j)}(x_i) \mid \in [0,1]$$
$$\Rightarrow 0 \leq \mid u_{F(p_j)}(x_i) - u_{G(p_j)}(x_i) \mid^q + \mid v_{F(p_j)}(x_i) - v_{G(p_j)}(x_i) \mid^q \leq 2 \text{ for all } i,j$$
$$\Rightarrow 0 \leq \frac{1}{2 \mid \mathrm{X} \mid\mid P \mid} \sum_{j=1}^{m} \sum_{i=1}^{n} (\mid u_{F(p_j)}(x_i) - u_{G(p_j)}(x_i) \mid^q + \mid v_{F(p_j)}(x_i) - v_{G(p_j)}(x_i) \mid^q) \leq 1$$
$$\Rightarrow 0 \leq \sqrt[q]{\frac{1}{2 \mid \mathrm{X} \mid\mid P \mid} \sum_{j=1}^{m} \sum_{i=1}^{n} (\mid u_{F(p_j)}(x_i) - u_{G(p_j)}(x_i) \mid^q + \mid v_{F(p_j)}(x_i) - v_{G(p_j)}(x_i) \mid^q)} \leq 1$$
$$\Rightarrow 0 \leq S((F,P),(G,P) \leq 1$$

(2) Since $\mid u_{F(p_j)}(x_i) - u_{G(p_j)}(x_i) \mid = \mid u_{G(p_j)}(x_i) - u_{F(p_j)}(x_i) \mid$ and $\mid v_{F(p_j)}(x_i) - v_{G(p_j)}(x_i) \mid = \mid v_{G(p_j)}(x_i) - v_{F(p_j)}(x_i) \mid$, $S((F,P),(G,P)) = S((G,P),(F,P))$.

(3) Since $(F,P) \subseteq (G,P) \subseteq (H,P)$, $u_{F(p_j)}(x_i) \leq u_{G(p_j)}(x_i) \leq u_{H(p_j)}(x_i)$ and $v_{F(p_j)}(x_i) \geq v_{G(p_j)}(x_i) \geq v_{H(p_j)}(x_i)$ for all i,j. Thus $\mid u_{F(p_j)}(x_i) - u_{G(p_j)}(x_i) \mid \leq \mid u_{F(p_j)}(x_i) - u_{H(p_j)}(x_i) \mid$, $\mid u_{G(p_j)}(x_i) - u_{H(p_j)}(x_i) \mid \leq \mid u_{F(p_j)}(x_i) - u_{H(p_j)}(x_i) \mid$ and $\mid v_{F(p_j)}(x_i) - v_{G(p_j)}(x_i) \mid \leq \mid v_{F(p_j)}(x_i) - v_{H(p_j)}(x_i) \mid$, $\mid v_{G(p_j)}(x_i) - v_{H(p_j)}(x_i) \mid \leq \mid v_{F(p_j)}(x_i) - v_{H(p_j)}(x_i) \mid$. As a result for all i,j,

$$\mid u_{F(p_j)}(x_i) - u_{G(p_j)}(x_i) \mid^q + \mid v_{F(p_j)}(x_i) - v_{G(p_j)}(x_i) \mid^q \leq \mid u_{F(p_j)}(x_i) - u_{H(p_j)}(x_i) \mid^q + \mid v_{F(p_j)}(x_i) - v_{H(p_j)}(x_i) \mid^q$$

$$\mid u_{G(p_j)}(x_i) - u_{H(p_j)}(x_i) \mid^q + \mid v_{G(p_j)}(x_i) - v_{H(p_j)}(x_i) \mid^q \leq \mid u_{F(p_j)}(x_i) - u_{H(p_j)}(x_i) \mid^q + \mid v_{F(p_j)}(x_i) - v_{H(p_j)}(x_i) \mid^q$$

which implies that

$$\sqrt[q]{\frac{1}{2 \mid \mathrm{X} \mid\mid P \mid} \sum_{j=1}^{m} \sum_{i=1}^{n} (\mid u_{F(p_j)}(x_i) - u_{G(p_j)}(x_i) \mid^q + \mid v_{F(p_j)}(x_i) - v_{G(p_j)}(x_i) \mid^q)} \leq \sqrt[q]{\frac{1}{2 \mid \mathrm{X} \mid\mid P \mid} \sum_{j=1}^{m} \sum_{i=1}^{n} (\mid u_{F(p_j)}(x_i) - u_{H(p_j)}(x_i) \mid^q + \mid v_{F(p_j)}(x_i) - v_{H(p_j)}(x_i) \mid^q)}$$

and

$$\sqrt[q]{\frac{1}{2 \mid \mathrm{X} \mid\mid P \mid} \sum_{j=1}^{m} \sum_{i=1}^{n} (\mid u_{G(p_j)}(x_i) - u_{H(p_j)}(x_i) \mid^q + \mid v_{G(p_j)}(x_i) - v_{H(p_j)}(x_i) \mid^q)} \leq \sqrt[q]{\frac{1}{2 \mid \mathrm{X} \mid\mid P \mid} \sum_{j=1}^{m} \sum_{i=1}^{n} (\mid u_{F(p_j)}(x_i) - u_{H(p_j)}(x_i) \mid^q + \mid v_{F(p_j)}(x_i) - v_{H(p_j)}(x_i) \mid^q)}$$

It follows that

$$1-\sqrt[q]{\frac{1}{2\mid \mathrm{X} \mid\mid P\mid}\sum_{j=1}^{m}\sum_{i=1}^{n}(\mid u_{F(p_j)}(x_i)-u_{G(p_j)}(x_i)\mid^q+\mid v_{F(p_j)}(x_i)-v_{G(p_j)}(x_i)\mid^q)}\geq$$

$$1-\sqrt[q]{\frac{1}{2\mid \mathrm{X} \mid\mid P\mid}\sum_{j=1}^{m}\sum_{i=1}^{n}(\mid u_{F(p_j)}(x_i)-u_{H(p_j)}(x_i)\mid^q+\mid v_{F(p_j)}(x_i)-v_{H(p_j)}(x_i)\mid^q)}$$

and

$$1-\sqrt[q]{\frac{1}{2\mid \mathrm{X} \mid\mid P\mid}\sum_{j=1}^{m}\sum_{i=1}^{n}(\mid u_{G(p_j)}(x_i)-u_{H(p_j)}(x_i)\mid^q+\mid v_{G(p_j)}(x_i)-v_{H(p_j)}(x_i)\mid^q)}\geq$$

$$1-\sqrt[q]{\frac{1}{2\mid \mathrm{X} \mid\mid P\mid}\sum_{j=1}^{m}\sum_{i=1}^{n}(\mid u_{F(p_j)}(x_i)-u_{H(p_j)}(x_i)\mid^q+\mid v_{F(p_j)}(x_i)-v_{H(p_j)}(x_i)\mid^q)}$$

This completes the proof. □

Example 2. *Consider two datasets on a universal set* $X=\{x_1,x_2,x_3,x_4\}$ *with a parameter set* $P=\{p_1,p_2,p_3,p_4,p_5\}$.

	p_1	p_2	p_3	p_4	p_5
x_1	$(0.5,0.2)$	$(0.7,0.2)$	$(0.8,0.1)$	$(0.9,0.1)$	$(0.7,0.2)$
x_2	$(0.8,0.1)$	$(0.6,0.2)$	$(0.7,0.1)$	$(0.8,0.1)$	$(0.6,0.2)$
x_3	$(0.7,0.5)$	$(0.7,0.1)$	$(0.8,0.1)$	$(0.8,0.2)$	$(0.7,0.2)$
x_4	$(0.6,0.2)$	$(0.5,0.2)$	$(0.5,0.1)$	$(0.3,0.5)$	$(0.5,0.2)$

	p_1	p_2	p_3	p_4	p_5
x_1	$(0.6,0.1)$	$(0.6,0.3)$	$(0.8,0.2)$	$(0.6,0.25)$	$(0.4,0.1)$
x_2	$(0.2,0.7)$	$(0.8,0.5)$	$(0.8,0.2)$	$(0.5,0.5)$	$(0.8,0.1)$
x_3	$(0.8,0.3)$	$(0.9,0.1)$	$(0.9,0.1)$	$(0.6,0.2)$	$(0.7,0.2)$
x_4	$(0.5,0.1)$	$(0.3,0.9)$	$(0.6,0.2)$	$(0.7,0.1)$	$(0.8,0.3)$

The optimum index for both datasets is $q_0=2$. *Hence, both the data sets are* $OFSS_q$ *for* $q\geq 2$. *Denoting them as* (F,P) *and* (G,P) *for* $q=2$, *the similarity measure between them, as per Theorem 2, is*

$$\begin{aligned}S((F,P),(G,P))&=1-\sqrt[2]{\frac{1}{2\times 4\times 5}\sum_{j=1}^{5}\sum_{i=1}^{4}(\mid u_{F(p_j)}(x_i)-u_{G(p_j)}(x_i)\mid^2+\mid v_{F(p_j)}(x_i)-v_{G(p_j)}(x_i)\mid^2)}\\&=0.7469\end{aligned}$$

3.3. Construction of Similarity measures for q-rung orthopair fuzzy soft sets

Now, we generate similarity measures for OFSS_qs, using the similarity measures of OFS_qs, which are constructed using fuzzy equivalences [67] and aggregation operations [68].

Definition 7. [68] *A function* $\tilde{f}:[0,1]\to[0,1]$ *is a fuzzy equivalence if it satisfies the following properties:*

(1) $\tilde{f}(x,y)=\tilde{f}(y,x)$ *for all* $x,y\in[0,1]$.

(2) $\tilde{f}(x,x)=1$ *for each* $x\in[0,1]$.

(3) $\tilde{f}(1,0)=0$.

(4) *For* $x,y,z\in[0,1]$, *if* $x\leq y\leq z$, *then* $\min\{\tilde{f}(x,y),\tilde{f}(y,z)\}\geq\tilde{f}(x,z)$

Definition 8. [68] *A function* $m:[0,1]^n\to[0,1]$ *is called an aggregation function if it satisfies the following:*

(1) $m(0, 0, \ldots, 0) = 0$, $m(1, 1, \ldots, 1) = 1$

(2) $m(x, x, \ldots, x) = x$

(3) *m is monotonically increasing in all of its arguments.*

Definition 9. [56] *Let X be a universal set, and $OFS_q(\mathrm{X})$ denotes the set of all q-rung orthopair fuzzy sets on X. Let F_1, F_2 and F_3 be OFS_qs in $OFS_q(\mathrm{X})$. A similarity measure is a mapping $S : OFS_q(\mathrm{X}) \times OFS_q(\mathrm{X}) \to [0, 1]$ satisfying all the following axioms:*

(1) $0 \leq S(F_1, F_2) \leq 1$.

(2) $S(F_1, F_2) = S(F_2, F_1)$

(3) $S(F_1, F_2) = 1$ *if and only if* $F_1 = F_2$

(4) $S(F_1, {F_1}^c) = 0$ *if and only if* F_1 *is a crisp set.*

(5) *If* $F_1 \subseteq F_2 \subseteq F_3$, *then* $S(F_1, F_3) \leq S(F_1, F_2)$ *and* $S(F_1, F_3) \leq S(F_2, F_3)$.

Theorem 3. *Let* $\mathrm{X} = \{x_1, x_2, \ldots, x_n\}$, $F_1 = \{\langle x, u_{F_1}(x_i), v_{F_1}(x_i)\rangle : x_i \in \mathrm{X}\}$, $F_2 = \{\langle x, u_{F_2}(x_i), v_{F_2}(x_i)\rangle : x_i \in \mathrm{X}\}$, $F_3 = \{\langle x, u_{F_3}(x_i), v_{F_3}(x_i)\rangle : x_i \in \mathrm{X}\}$ *be three OFS_qs on X. Let $\tilde{f}$ be a fuzzy equivalence, and m_1, m_2 be two aggregation functions. Then*

$$S(F_1, F_2) = m_1[m_2\{\tilde{f}(u_{F_1}(x_1), u_{F_2}(x_1)), \tilde{f}(v_{F_1}(x_1), v_{F_2}(x_1))\}, m_2\{\tilde{f}(u_{F_1}(x_2), u_{F_2}(x_2)), \tilde{f}(v_{F_1}(x_1), v_{F_2}(x_1))\}, \\ \ldots\ldots, m_2\{\tilde{f}(u_{F_n}(x_1), u_{F_2}(x_n), \tilde{f}(v_{F_1}(x_n), v_{F_2}(x_n))\}] \quad (4)$$

is a similarity measure for OFS_qs.

Proof: We check the axioms for a similarity measure of OFS_qs listed in Definition 9.

(1) Since fuzzy equivalence and aggregation operations give values in $[0, 1]$ we get $0 \leq S(F_1, F_2) \leq 1$.

(2) By condition (1) of fuzzy equivalence $\tilde{f}$ in Definition 7, $S(F_1, F_2) = S(F_2, F_1)$

(3) We find,
$$\begin{aligned} F_1 = F_2 &\iff u_{F_1}(x_i) = u_{F_2}(x_i), v_{F_1}(x_i) = v_{F_2}(x_i) \,\text{for all}\, x_i \in \mathrm{X} \\ &\iff \tilde{f}(u_{F_1}(x_i), u_{F_2}(x_i) = \tilde{f}(v_{F_1}(x_i), v_{F_2}(x_i)) = 1 \,\text{for all}\, x_i \in \mathrm{X} \\ &\iff m_2\{\tilde{f}(u_{F_1}(x_i), u_{F_2}(x_i), \tilde{f}(v_{F_1}(x_i), v_{F_2}(x_i))\} = 1 \,\text{for all}\, x_i \in \mathrm{X} \\ &\iff S(F_1, F_2) = 1 \end{aligned}$$

(4) We have
$$\begin{aligned} F_1 \,\text{is a crisp set} &\iff (u_{F_1}(x_i), v_{F_1}(x_i)) = (1, 0) \,\text{or}\, (0, 1) \\ &\iff (u_{F_1}(x_i), u_{F_1^c}(x_i)) = (1, 0) \,\text{or}\, (0, 1) \,\text{and}\, (v_{F_1}(x_i), v_{F_1^c}(x_i)) = (1, 0) \,\text{or}\, (0, 1) \,\text{for all}\, x_i \\ &\iff \tilde{f}(u_{F_1}(x_i), u_{F_1^c}(x_i)) = \tilde{f}(v_{F_1}(x_i), v_{F_1^c}(x_i)) = 0 \,\text{for all}\, x_i \\ &\iff m_2\{\tilde{f}(u_{F_1}(x_i), u_{F_1^c}(x_i), \tilde{f}(v_{F_1}(x_i), v_{F_1^c}(x_i))\} = 0 \,\text{for all}\, x_i \in \mathrm{X} \\ &\iff S(F_1, F_1^c) = 0 \end{aligned}$$

(5) Suppose $F_1 \subseteq F_2 \subseteq F_3$

$$\begin{aligned}F_1 \subseteq F_2 \subseteq F_3 \Rightarrow\ & u_{F_1}(x_i) \leq u_{F_2}(x_i) \leq u_{F_3}(x_i) \,\text{and}\, v_{F_1}(x_i) \geq v_{F_2}(x_i) \geq v_{F_3}(x_i) \,\text{for all}\, x_i \in \mathrm{X}\\ &\Rightarrow \tilde{f}(u_{F_1}(x_i), u_{F_3}(x_i)) \leq \tilde{f}(u_{F_1}(x_i), u_{F_2}(x_i)), \tilde{f}(v_{F_1}(x_i), v_{F_3}(x_i)) \leq \tilde{f}(v_{F_1}(x_i), v_{F_2}(x_i))\\ &\qquad \text{for all}\, x_i \in \mathrm{X}\, (\text{by (4) in Definition 7})\\ &\Rightarrow m_2\{\tilde{f}(u_{F_1}(x_i), u_{F_3}(x_i), \tilde{f}(v_{F_1}(x_i), v_{F_3}(x_i))\} \leq m_2\{\tilde{f}(u_{F_1}(x_i), u_{F_2}(x_i), \tilde{f}(v_{F_1}(x_i), v_{F_2}(x_i))\}\\ &\qquad \text{for all}\, x_i \in \mathrm{X}\, (\text{by (3) in Definition 8 for}\, m_2)\\ &\Rightarrow S(F_1, F_3) \leq S(F_1, F_2)\, (\text{by (3) in Definition 8 for}\, m_1)\end{aligned}$$

Similarly, we can show that $S(F_1, F_3) \leq S(F_2, F_3)$.

Hence equation (4) gives a similarity measure for OFS_qs. □

To construct similarity measures for OFSS_qs, we adopt the approach of aggregating the similarity measures between the corresponding OFS_qs for each parameter.

Theorem 4. *Let $\mathrm{X} = \{x_1, x_2, \ldots, x_n\}$ be the universal set, and $P = \{p_1, p_2, \ldots. p_m\}$ be the set of parameters. Let (G, P), (H, P) and (I, P) be three $OFSS_q$s in $OFSS_q(\mathrm{X}, P)$. Let F_j, F'_j and F''_j denote the OFS_qs corresponding to p_j in (G, P) (H, P) and (I, P), respectively. Let S denote a similarity measure defined for OFS_qs, and let and m be an aggregation function, then*

$$\tilde{S}((G, P), (H, P)) = m(S(F_1, F'_1), S(F_2, F'_2), \ldots\ldots, S(F_m, F'_m)) \tag{5}$$

is a similarity measure for $OFSS_q$s.

Proof: To check the axioms for a similarity measure of OFSS_q listed in Definition 6.

(1) Since an aggregation function take values in $[0, 1]$, $0 \leq \tilde{S}((G, P), (H, P)) \leq 1$.

(2) We have

$$\begin{aligned}\tilde{S}((G, P), (H, P)) &= m(S(F_1, F'_1), S(F_2, F'_2), \ldots, S(F_m, F'_m))\\ &= m(S(F'_1, F_1), S(F'_2, F_2), \ldots, S(F'_m, F_m))\, (\text{by condition (2) in Definition 9})\\ &= \tilde{S}((H, P), (G, P))\end{aligned}$$

(3) We have,

$$\begin{aligned}(G, P) = (H, P) &\iff F_j = F'_j \,\text{for all}\, p_j \in P\\ &\iff S(F_j, F'_j) = 1 \,\text{for all}\, j\\ &\iff \tilde{S}((G, P), (H, P)) = 1\end{aligned}$$

(4) Suppose $(G, P) \subseteq (H, P) \subseteq (I, P)$

$$\begin{aligned}(G, P) \subseteq (H, P) \subseteq (I, P) \Rightarrow\ & F_j \subseteq F'_j \subseteq F''_j \,\text{for each}\, j\\ &\Rightarrow S(F_j, F''_j) \leq S(F_j, F'_j) \,\text{for each}\, j\, (\text{by (4) in Definition 9})\\ &\Rightarrow m(S(F_1, F''_1), S(F_2, F''_2), \ldots\ldots, S(F_m, F''_m)) \leq m(S(F_1, F'_1), S(F_2, F'_2), \ldots, S(F_m, F'_m))\\ &\qquad (\text{by (3) in Definition 8 for}\, m)\\ &\Rightarrow \tilde{S}((G, P), (I, P)) \leq \tilde{S}((G, P), (H, P))\end{aligned}$$

Similarly, we can show $\tilde{S}((G, P), (I, P)) \leq \tilde{S}((H, P), (I, P))$

Hence equation (5) gives a similarity measure for OFSS_qs. □

Example 3. *For fuzzy equivalence $\tilde{f} = 1 - \mid x - y \mid$, and the aggregation function $m = m_1 = m_2 = 1 - (\frac{1}{n}\sum_{i=1}^{n}(1 - x_i^q))^{\frac{1}{q}}$, the obtained similarity measure is the one mentioned in equation (3).*

Various expressions for the similarity measure of OFSS_qs can be formulated by selecting different fuzzy equivalences and different aggregation functions. As indicated in [56], similarity-based entropy measures can be constructed for OFS_qs, and this concept can similarly be applied to OFSS_qs. For $(F,P) \in \mathrm{OFSS}_q(\mathrm{X},P)$, $E(F,P) = S((F,P),(F,P)^c)$. Note that the entropy measure in equation (2) can be expressed as a similarity-based entropy by using the similarity measure in equation (3).

4|Applications of Entropy and Similarity Measures in Decision making

This section elaborates on the utilization of entropy and similarity measures of OFSS_qs in real-life situations. The parameter q governs these measures for OFSS_qs, and its value can be adjusted, enabling efficient handling of inputs in decision making problems. These measures can potentially be applied to decision making problems, particularly when there are dubious inputs and a quick decision needs to be made. Specifically, when dealing with dubious inputs, an appropriate measure can be selected based on the user's requirements. These measures, controlled by the parameter q allow for adjustment at any time by merely changing the value of q to manage dubious inputs effectively. In this section, we present two real-life hypothetical scenarios, each illustrating the potential application of an entropy and a similarity measure, respectively, using hypothetical datasets in orthopairs.

4.1|The scenario for entropy measure

Background of the scenario: Suppose people crossing a checkpoint are screened for combating the spread of Covid-19. Let there are four persons, A, B, C, D. The 4 persons are assessed according to three criteria: x_1: proximity with Covid-19 outbreak locations
x_2: proximity with verified Covid-19 positive persons
x_3: proximity with family members.
For each criterion, three methods are used to access their route of travel:
p_1: GPS
p_2: QR code scanning apps
p_3: CCTV at public transport stations
It is to determine which among the 4 should undergo further testing for Covid-19.

The evaluations of persons in orthopairs is given in Tables 1-4

Table 1. Person A

	p_1	p_2	p_3
x_1	$u_{p_1}(x_1) = 0.1$ $v_{p_1}(x_1) = 0.9$	$u_{p_2}(x_1) = 0.0$ $v_{p_2}(x_1) = 0.9$	$u_{p_3}(x_1) = 0.0$ $v_{p_3}(x_1) = 0.8$
x_2	$u_{p_1}(x_2) = 0.2$ $v_{p_1}(x_2) = 0.9$	$u_{p_2}(x_2) = 0.1$ $v_{p_2}(x_2) = 0.9$	$u_{p_3}(x_2) = 0.2$ $v_{p_3}(x_2) = 0.8$
x_3	$u_{p_1}(x_3) = 0.1$ $v_{p_1}(x_3) = 0.9$	$u_{p_2}(x_2) = 0.0$ $v_{p_2}(x_3) = 1.0$	$u_{p_3}(x_3) = 0.0$ $v_{p_3}(x_3) = 1.0$

To illustrate the evaluations, consider person C as an example. For the person C, $u_{p_3}(x_1) = 0.5$ represents that the person is 50% certain to have been in close proximity with Covid-19 outbreak locations (x_1), according to surveillance through CCTV at public transport stations (p_3). On the other hand, $v_{p_3}(x_1) = 0.2$, represents that person C is 20% certain to have stayed away from Covid-19 outbreak locations (x_1), according to the surveillance through CCTV at public transport stations (p_3). Note that for the person C, $u_{p_1}(x_1) = 0.5$ and $v_{p_1}(x_1) = 0.6$, which gives $u_{p_1}(x_1) + v_{p_1}(x_1) > 1$, indicating the existence of contradicting information.

Table 2. Person B

	p_1	p_2	p_3
x_1	$u_{p_1}(x_1) = 1.0$ $v_{p_1}(x_1) = 0.0$	$u_{p_2}(x_1) = 0.8$ $v_{p_2}(x_1) = 0.0$	$u_{p_3}(x_1) = 0.9$ $v_{p_3}(x_1) = 0.1$
x_2	$u_{p_1}(x_2) = 0.9$ $v_{p_1}(x_2) = 0.0$	$u_{p_2}(x_2) = 0.9$ $v_{p_2}(x_2) = 0.1$	$u_{p_3}(x_2) = 1.0$ $v_{p_3}(x_2) = 0.0$
x_3	$u_{p_1}(x_3) = 0.9$ $v_{p_1}(x_3) = 0.1$	$u_{p_2}(x_2) = 1.0$ $v_{p_2}(x_3) = 0.0$	$u_{p_3}(x_3) = 0.9$ $v_{p_3}(x_3) = 0.0$

Table 3. Person C

	p_1	p_2	p_3
x_1	$u_{p_1}(x_1) = 0.5$ $v_{p_1}(x_1) = 0.6$	$u_{p_2}(x_1) = 0.3$ $v_{p_2}(x_1) = 0.6$	$u_{p_3}(x_1) = 0.5$ $v_{p_3}(x_1) = 0.2$
x_2	$u_{p_1}(x_2) = 0.4$ $v_{p_1}(x_2) = 0.3$	$u_{p_2}(x_2) = 0.3$ $v_{p_2}(x_2) = 0.1$	$u_{p_3}(x_2) = 0.2$ $v_{p_3}(x_2) = 0.2$
x_3	$u_{p_1}(x_3) = 0.4$ $v_{p_1}(x_3) = 0.1$	$u_{p_2}(x_2) = 0.5$ $v_{p_2}(x_3) = 0.7$	$u_{p_3}(x_3) = 0.4$ $v_{p_3}(x_3) = 0.3$

Table 4. Person D

	p_1	p_2	p_3
x_1	$u_{p_1}(x_1) = 0.9$ $v_{p_1}(x_1) = 0.1$	$u_{p_2}(x_1) = 0.5$ $v_{p_2}(x_1) = 0.5$	$u_{p_3}(x_1) = 0.7$ $v_{p_3}(x_1) = 0.6$
x_2	$u_{p_1}(x_2) = 0.8$ $v_{p_1}(x_2) = 0.8$	$u_{p_2}(x_2) = 0.7$ $v_{p_2}(x_2) = 0.6$	$u_{p_3}(x_2) = 0.4$ $v_{p_3}(x_2) = 0.4$
x_3	$u_{p_1}(x_3) = 0.9$ $v_{p_1}(x_3) = 0.8$	$u_{p_2}(x_2) = 0.6$ $v_{p_2}(x_3) = 0.1$	$u_{p_3}(x_3) = 0.7$ $v_{p_3}(x_3) = 0.0$

As the persons pass through the checkpoint, if their cases are clear-cut, then they will either be given clearance, or taken for quarantine. The issue arises when a non-clear-cut case appear, and in such cases further Covid-19 tests such as PCR will be desired.

Formation of OFSS$_q$s for entropy measures through choice of q: If a chief executive (decision-maker) is very strict with the clarity of the data, he would choose $q = 1$, resulting in the condition $0 \leq u_p(x) + v_p(x) \leq 1$ for all $p \in P$ and $x \in$ X. As $u_{p_1}(x_1) + v_{p_1}(x_1) > 1$ for person C, the data is deemed faulty, and a re-check of the surveillance system will be carried out. On the other hand, when time is at stake and quick decisions must be made, the chief executive may increase q as needed to allow for some degree of contradiction of information. This in turn, permits the formation of four OFSS$_q$, (A, P), (B, P), (C, P) and (D, P), with $P = \{p_1, p_2, p_3\}$ and $X = \{x_1, x_2, x_3\}$, allowing for the calculation of entropy.

The optimum index q_0 for the dataset of each person is found, with the maximum among the four indices being 5. Therefore, the datasets are considered OFSS$_q$s for $q \geq 5$. The values of the entropy measure for each OFSS$_q$ are shown in Table 5.

Table 5. Entropy values of each person

q	5	6	7	8	9	10	11	12	13	14	15
$E_q(A, P)$	0.1436	0.1360	0.1288	0.1221	0.1157	0.1098	0.1043	0.0991	0.0943	0.0898	0.0856
$E_q(B, P)$	0.0941	0.0900	0.0860	0.0822	0.0785	0.0750	0.0717	0.0685	0.0655	0.0627	0.0601
$E_q(C, P)$	0.7530	0.7466	0.7415	0.7372	0.7337	0.7307	0.7282	0.7261	0.7242	0.7226	0.7211
$E_q(D, P)$	0.4331	0.4060	0.3849	0.3679	0.3539	0.3422	0.3321	0.3234	0.3158	0.3091	0.3031

Thus, both C and D have higher entropy than A and B. However, it is observed that $E_q(D,P)$ is significantly affected by the choice of q. In conclusion, C and D should undergo further Covid-19 tests, such as PCR; however, the order of conducting the tests (whether C or D first) will be subject to preferences.

Comparative Analysis: To compare the proposed entropy measure for OFSS$_q$s, the entropy measure provided in [43] is used, which is expressed as:

$$E(F,P)=\frac{1}{\sqrt{2}-1}\sum_{j=1}^{m}\sum_{i=1}^{n}\left[sin\left(\frac{\pi(1+(u_{F(p_j)}(x_i))^q-(v_{F(p_j)}(x_i))^q)}{4}\right)+sin\left(\frac{\pi(1-(u_{F(p_j)}(x_i))^q+(v_{F(p_j)}(x_i))^q)}{4}\right)-1\right] \tag{6}$$

By applying the measure from equation (6) in the context described in section 4.1to calculate the entropy values for each person, the numerical values in Table 6 are obtained for q ranging from 5to 15.

Table 6. Entropy values of each person using the entropy measure in [43]

q	5	6	7	8	9	10	11	12	13	14	15
$E_q(A,P)$	4.9726	5.3903	5.7172	5.9744	6.1779	6.3396	6.4686	6.5718	6.6546	6.7211	6.7747
$E_q(B,P)$	4.0844	4.4623	4.7634	5.0040	5.1969	5.3517	5.4764	5.5768	5.6578	5.7232	5.7760
$E_q(C,P)$	8.9708	8.9855	8.9928	8.9965	8.9983	8.9992	8.9996	8.9998	8.9999	9.0000	9.0000
$E_q(D,P)$	8.5137	8.6033	8.6721	8.7270	8.7721	8.8097	8.8413	8.8679	8.8902	8.9090	8.9248

Here, both persons C and D have higher entropy than A and B. This observation supports the idea that the proposed measure is compatible with the existing one. However, unlike the entropy measure introduced in this article, it is found that $E_q(A,P)$ and $E_q(B,P)$ are more affected by the choice of q when using the measure in [43].

4.2|The Scenario for similarity measure

Background of the scenario: Suppose there are three medical experts p_1, p_2, p_3, in a clinic. In their opinion, there are 6 symptoms to be considered for distinguishing between two diseases (A: Covid-19, B: Common Flu), which are
x_1: difficulty in breathing
x_2: loss of taste
x_3: chest pain
x_4: running nose
x_5: vomiting
x_6: having chills
Based on their previous experience with patients exhibiting these symptoms and their subsequent diagnoses, each expert provides their evaluations of these symptoms for both diseases. The presence of these six symptoms for both diseases is outlined in Tables 7 and 8.

Table 7. A: Covid-19

	x_1	x_2	x_3	x_4	x_5	x_6
p_1	$u_{p_1}(x_1)=0.8$	$u_{p_1}(x_2)=0.9$	$u_{p_1}(x_3)=0.7$	$u_{p_1}(x_4)=0.5$	$u_{p_1}(x_5)=0.1$	$u_{p_1}(x_6)=0.2$
	$v_{p_1}(x_1)=0.2$	$v_{p_1}(x_2)=0.1$	$v_{p_1}(x_3)=0.2$	$v_{p_1}(x_4)=0.2$	$v_{p_1}(x_5)=0.8$	$v_{p_1}(x_6)=0.5$
p_2	$u_{p_2}(x_1)=0.8$	$u_{p_2}(x_2)=0.8$	$u_{p_2}(x_3)=0.4$	$u_{p_2}(x_4)=0.5$	$u_{p_2}(x_5)=0.1$	$u_{p_2}(x_6)=0.1$
	$v_{p_2}(x_1)=0.1$	$v_{p_2}(x_2)=0.3$	$v_{p_2}(x_3)=0.5$	$v_{p_2}(x_4)=0.2$	$v_{p_1}(x_5)=0.8$	$v_{p_2}(x_6)=0.8$
p_3	$u_{p_3}(x_1)=0.8$	$u_{p_3}(x_2)=0.9$	$u_{p_3}(x_3)=0.7$	$u_{p_3}(x_4)=0.5$	$u_{p_3}(x_5)=0.0$	$u_{p_3}(x_6)=0.0$
	$v_{p_3}(x_1)=0.3$	$v_{p_3}(x_2)=0.4$	$v_{p_3}(x_3)=0.1$	$v_{p_3}(x_4)=0.2$	$v_{p_3}(x_5)=1.0$	$v_{p_3}(x_6)=1.0$

In the case of disease A, $u_{p_1}(x_4)=0.5$ and $v_{p_1}(x_4)=0.2$ indicate that, according to expert p_1, there is a 50% certainty that a patient with the disease will exhibit symptom x_4, and a 20% certainty that the patient will not exhibit the symptom.

Table 8. B: Common Flu

	x_1	x_2	x_3	x_4	x_5	x_6
p_1	$u_{p_1}(x_1)=0.9$ $v_{p_1}(x_1)=0.2$	$u_{p_1}(x_2)=0.1$ $v_{p_1}(x_2)=0.9$	$u_{p_1}(x_3)=0.5$ $v_{p_1}(x_3)=0.5$	$u_{p_1}(x_4)=0.8$ $v_{p_1}(x_4)=0.1$	$u_{p_1}(x_5)=0.1$ $v_{p_1}(x_5)=0.9$	$u_{p_1}(x_6)=0.5$ $v_{p_1}(x_6)=0.5$
p_2	$u_{p_2}(x_1)=0.9$ $v_{p_2}(x_1)=0.1$	$u_{p_2}(x_2)=0.0$ $v_{p_2}(x_2)=1.0$	$u_{p_2}(x_3)=0.2$ $v_{p_2}(x_3)=0.6$	$u_{p_2}(x_4)=1.0$ $v_{p_2}(x_4)=0.0$	$u_{p_2}(x_5)=0.1$ $v_{p_1}(x_5)=0.8$	$u_{p_2}(x_6)=0.1$ $v_{p_2}(x_6)=0.6$
p_3	$u_{p_3}(x_1)=0.8$ $v_{p_3}(x_1)=0.2$	$u_{p_3}(x_2)=0.1$ $v_{p_3}(x_2)=0.8$	$u_{p_3}(x_3)=0.2$ $v_{p_3}(x_3)=0.8$	$u_{p_3}(x_4)=0.8$ $v_{p_3}(x_4)=0.3$	$u_{p_3}(x_5)=0.1$ $v_{p_3}(x_5)=0.9$	$u_{p_3}(x_6)=0.1$ $v_{p_3}(x_6)=0.7$

In the case of disease B, $u_{p_1}(x_4) = 0.8$ and $v_{p_1}(x_4) = 0.1$ indicate that, in the opinion of expert p_1, there is 80% certainty that a patient with disease B will exhibit symptom x_4, and a 10% certainty that the patient will not. Note that $u_{p_1}(x_1) = 0.2$ and $v_{p_1}(x_1) = 0.9$, which implies $u_{p_1}(x_1) + v_{p_1}(x_1) > 1$, indicating the existence of conflicting information for $q = 1$.

Now, two patients, C and D, are coming to the clinic. After the medical examination, the three medical experts inferred the evaluations of their symptoms, as shown in Tables 9 and 10.

Table 9. C: Patient

	x_1	x_2	x_3	x_4	x_5	x_6
p_1	$u_{p_1}(x_1)=0.7$ $v_{p_1}(x_1)=0.2$	$u_{p_1}(x_2)=0.4$ $v_{p_1}(x_2)=0.5$	$u_{p_1}(x_3)=0.5$ $v_{p_1}(x_3)=0.4$	$u_{p_1}(x_4)=0.6$ $v_{p_1}(x_4)=0.1$	$u_{p_1}(x_5)=0.3$ $v_{p_1}(x_5)=0.7$	$u_{p_1}(x_6)=0.4$ $v_{p_1}(x_6)=0.3$
p_2	$u_{p_2}(x_1)=0.5$ $v_{p_2}(x_1)=0.5$	$u_{p_2}(x_2)=0.5$ $v_{p_2}(x_2)=0.1$	$u_{p_2}(x_3)=0.3$ $v_{p_2}(x_3)=0.7$	$u_{p_2}(x_4)=0.2$ $v_{p_2}(x_4)=0.2$	$u_{p_2}(x_5)=0.4$ $v_{p_1}(x_5)=0.4$	$u_{p_2}(x_6)=0.5$ $v_{p_2}(x_6)=0.3$
p_3	$u_{p_3}(x_1)=0.6$ $v_{p_3}(x_1)=0.2$	$u_{p_3}(x_2)=0.5$ $v_{p_3}(x_2)=0.1$	$u_{p_3}(x_3)=0.4$ $v_{p_3}(x_3)=0.6$	$u_{p_3}(x_4)=0.5$ $v_{p_3}(x_4)=0.3$	$u_{p_3}(x_5)=0.3$ $v_{p_3}(x_5)=0.2$	$u_{p_3}(x_6)=0.4$ $v_{p_3}(x_6)=0.4$

For the patient C, $u_{p_1}(x_1) = 0.7$, $v_{p_1}(x_1) = 0.2$ denote that according to the expert p_1, there is 70% certainty that the patient C has difficulty in breathing, and 20% certain that he does not have difficulty in taking breath.

Table 10. D: Patient

	x_1	x_2	x_3	x_4	x_5	x_6
p_1	$u_{p_1}(x_1)=0.8$ $v_{p_1}(x_1)=0.0$	$u_{p_1}(x_2)=0.2$ $v_{p_1}(x_2)=0.7$	$u_{p_1}(x_3)=0.4$ $v_{p_1}(x_3)=0.4$	$u_{p_1}(x_4)=0.6$ $v_{p_1}(x_4)=0.2$	$u_{p_1}(x_5)=0.1$ $v_{p_1}(x_5)=0.9$	$u_{p_1}(x_6)=0.5$ $v_{p_1}(x_6)=0.4$
p_2	$u_{p_2}(x_1)=0.8$ $v_{p_2}(x_1)=0.6$	$u_{p_2}(x_2)=0.1$ $v_{p_2}(x_2)=0.9$	$u_{p_2}(x_3)=0.3$ $v_{p_2}(x_3)=0.7$	$u_{p_2}(x_4)=1.0$ $v_{p_2}(x_4)=0.0$	$u_{p_2}(x_5)=0.0$ $v_{p_1}(x_5)=0.8$	$u_{p_2}(x_6)=0.2$ $v_{p_2}(x_6)=0.5$
p_3	$u_{p_3}(x_1)=0.9$ $v_{p_3}(x_1)=0.0$	$u_{p_3}(x_2)=0.0$ $v_{p_3}(x_2)=1.0$	$u_{p_3}(x_3)=0.2$ $v_{p_3}(x_3)=0.5$	$u_{p_3}(x_4)=0.6$ $v_{p_3}(x_4)=0.4$	$u_{p_3}(x_5)=0.2$ $v_{p_3}(x_5)=0.9$	$u_{p_3}(x_6)=0.4$ $v_{p_3}(x_6)=0.7$

Note that in the case of patient D, $u_{p_2}(x_1) = 0.8$ and $v_{p_2}(x_1) = 0.6$, which implies $u_{p_2}(x_1) + v_{p_2}(x_1) > 1$. This shows the existence of contradicting information. Hence, patient C represent someone with a consistent result of diagnosis, whereas patient D represents someone with dubious diagnostic results. We need to determine whether these patients should undergo additional PCR testing, especially when the indications of Covid-19 are sufficiently notable.

Formation of OFSS$_q$s for similarity measures through choice of q: If the chief expert is very strict with the clarity of the data, he would choose $q = 1$, causing the condition $0 \leq u_p(x) + v_p(x) \leq 1$ for all p and x. However, since $u_{p_1}(x_1) + v_{p_1}(x_1) > 1$ for patient D, the data is deemed faulty, and a re-diagnosis of the patient will have to be carried out. On the other hand, when time is at stake, and quick decisions need to be made, the chief expert may increase q as appropriate to allow some degree of contradiction in information. This consequently permits the formation of four OFSS$_q$s as $(A, P), (B, P), (C, P)$, (D, P), along with the calculation of their similarity measures.

The optimum index for the data sets of both the disease and patients are found: q_0 for A and C is 1 and for B and D is 2. Hence the data sets of A, B, C and D are OFSS_qs for $q \geq 2$. By applying the the formula in Theorem 2, the numerical values in Table 11 are obtained for q ranging from 2 to 10.

TABLE 11. Similarity measure values of patients C and D to diseases A and B

q	2	3	4	5	6	7	8	9	10
$S_q((A,P),(C,P)$	0.6781	0.6352	0.5969	0.5620	0.5304	0.5022	0.4773	0.4553	0.4361
$S_q((B,P),(C,P)$	0.6138	0.5546	0.5045	0.4640	0.4317	0.4059	0.3850	0.3678	0.3535
$S_q((A,P),(D,P)$	0.6972	0.6192	0.5573	0.5078	0.4676	0.4341	0.4059	0.3819	0.3611
$S_q((B,P),(D,P)$	0.8463	0.8105	0.7763	0.7458	0.7196	0.6976	0.6791	0.6634	0.6501

From the calculations of similarity measures, it is found that patient C should undergo further Covid-19 tests, such as PCR, as $S_q((A,P),(C,P)) > S_q((B,P),(C,P))$ for all values the assigned to q in this particular problem. In contrast, for the patient D, $S_q((A,P),(D,P)) < S_q((B,P),(D,P))$ for all values of q. Thus, the decision is made that patient D will not undergo a Covid-19 test.

In the real-life contexts discussed, the choice of q will determine how much tolerance is given to the occurrence of contradictory information. This will determine whether or not the OFSS_qs will be formed and whether the further computations for similarity and entropy measures are meaningful enough to proceed. Similarity and entropy measures are studied for their effects on handling both consistent and dubious inputs. In particular, when dealing with dubious inputs, a suitable similarity and entropy measure may be chosen based on the user's requirements. These measures, governed by a parameter q permit adjustment at any instant, as only the value of q needs to be changed to produce different ways of handling dubious inputs. Furthermore, a single orthopair in the dataset can influence the selection of a suitable value of q that satisfies the entire dataset. If such a choice results in selecting a large value of q, it could be reflected in the calculated numerical values of information measures such as entropy and similarity measures.

5|Conclusion

This study demonstrates the practical importance of q-rung orthopair fuzzy soft sets (OFSS_q) and highlights their benefits compared to intuitionistic fuzzy soft sets (IFSS) in managing real-world data. When datasets are very large, the absence of contradictory information cannot be guaranteed. Such contradictory information, although much rarer than normal data, becomes significantly threatening when the dataset is large. Then, the axioms for entropy measures and similarity measures for OFSS_qs are established, and are applied to decision making problems related to the spread of Covid-19. This study lacks a comprehensive theoretical framework for constructing entropy measures for OFSS_qs and provides only a limited set of expressions for entropy and similarity measures. The entropy measures presented in the article primarily focus on the fuzziness entropy of OFSS_qs, defined by the difference between the membership and non-membership degrees. However, the study overlooks another key aspect-hesitation entropy-characterized by the hesitation degree of OFSS_qs. To enhance the entropy measures of OFSS_qs, it is essential to incorporate both fuzziness entropy and hesitation entropy through the introduction of total entropy measures for OFSS_qs. A potential future research direction involves developing additional expressions for entropy, similarity measures, and other information measures within q-rung orthopair fuzzy soft framework. Additionally, further research could explore the possibility of developing these measures using aggregation operators like t-norms and t-conorms. Although OFSS_qs offer advantages over IFSSs and PFSSs, they assign equal importance to both membership and non-membership values, which may not be necessary in certain real-life scenarios. Therefore, research could also focus on exploring the information measures for generalizations of OFSS_qs, such as (a,b)-fuzzy soft sets [69].

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Author Contribution

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Data Availability

All data supporting the reported findings in this research paper are provided within the manuscript.

Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings.

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